# 1 Exordium

Second order differential equations are an awesome subset of differential equations. The familiar Newton's second law $F(x,t) = \frac{d^2x}{dt^2}$ with the modification of drag $b$ and spring constant $c$ is often, most usefully, cast as a second order differential equation, $F(x,t) = a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx$. Moreover, there's the equation for charge $Q = Q(t)$ in the LRC-circuit equation, $L\frac{d^2Q}{dt^2} + Q/C + \frac{dQ}{dt}R = V(t)$.

Before I go over the usual methods of solution for second order differential equations, I'd like to introduce to you the unifying concept behind such equations—especially, the homogenous differential equation. It's really all about a very simple equation,

$$\hat{H}y = Ey, \tag{1}$$

where $y = y(t)$, $\hat{H}$ is a linear operator, and $E = E(t)$.

In the $\boxed{\text{second order case}}$,

$$\begin{aligned} \hat{H} &= \frac{d^2}{dt^2} + p(t)\frac{d}{dt} && (2) \\ E &= -q(t). && (3) \end{aligned}$$

$\boxed{\text{The problem, then, is to find all } y \text{ for which the equation } \hat{H}y = Ey \text{ is true.}}$

# 2 Homogenous Equations with Constant Coefficients

## 2.1 Distinct Roots

If we're lucky enough to have a homogenous differential equation,

$$Hy = Ey \Rightarrow \left(\frac{d^2}{dt^2} + p\frac{d}{dt}\right) y = qy, \tag{4}$$

involving $p = p(t) = constant$ and $q = q(t) = constant$, then we can begin with an ansatz—an educated guess:

$$y(t) = e^{\alpha t} \tag{5}$$

Now, we plug in this ansatz into *Equation 1*, using *Equation 3* for $\hat{H}$, to get

$$\begin{aligned} \hat{H}y &= (\alpha^2 + p\alpha)y && (6) \\ Ey &= qy. && (7) \end{aligned}$$

The right side of the arrow sign gives us a boring ol'algebra equation, and assuming that $y \neq 0$, we have, simply

$$\alpha^2 + p\alpha = q \Rightarrow \alpha^2 + p\alpha - q = 0. \tag{8}$$

The problem now is to solve for $\alpha$. If you're lucky, you can factor to solve for $\alpha$. But, if not, the other brain-less solution is just to apply the quadratic equation (to wit: $ax^2+bx+c = 0 \Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$) to get

$$\alpha = \frac{-p^2}{2} \pm \frac{\sqrt{p^2+4q}}{2}. \tag{9}$$

Note that the quadratic equation always yields two solutions to a problem. But, a complication arises if we have both roots the same. Namely, from the naive method we have just developed, there would only be one exponential solution to our second order differential equation. This would mean that, suppose we apply the initial conditions,

$$Hy = Ey \quad y(t_0) = y_0 \ \ and \ \ y'(t_0) = y'_0. \tag{10}$$

to the solution $y(t) = Ae^{\alpha t}$. We would end up with the strict constraint that $y(t_0) = Ae^{\alpha t_0} = y_0$ and $y'(t_0) = A\alpha e^{\alpha t_0} = y'_0$. This is rarely, if ever, true. The one case where it is true is the trivial case, where $Hy = Ey \ \ y(t_0) = 0 = y'(t_0)$, since $y(t) = 0$.

#### 2.1.1 $c_1$, $c_2$ and the Wronskian

So, in general you have two different solutions $y_1$ and $y_2$ for your second order differential equation, *Equation 3*.

Now, any combination of those two solutions gives you another solution. Because the combination is linear, such a combination is called a *linear combination*,

$$y_{general}(t) = c_1 y_1(t) + c_2 y_2(t). \tag{11}$$

You can check that $y_{general}$ gives another solution by plugging it into *Equation 3*.

Given two initial conditions, $y(t_0) = y_0$ and $y(t_0) = y_0^{'}$, there is a way to find $c_1$ and $c_2$—as well as a way to find whether they exist. Suppose $y_1(t)$ and $y_2(t)$ are solutions to *Equation 3*. Then, given the form $y_{general} = c_1 y_1(t) + c_2 y_2(t)$ and the aforementioned initial conditions, the coefficients are determined by the usual method of determinants for solving regular ol' linear equations,[1]

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0^{'} & y_2^{'}(t_0) \end{vmatrix}}{W(t_0)} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1^{'}(t_0) & y_0^{'} \end{vmatrix}}{W(t_0)}, \tag{16}$$

where to make the equation less imposing, we have defined the denominator to be $W(t_0)$,

$$W(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1^{'}(t_0) & y_2^{'}(t_0) \end{vmatrix} \tag{17}$$

W is also known as the *Wronskian.*

The set of solutions $y_1(t)$ and $y_2(t)$ where $W(t) \neq 0$ forms a *fundamental set of solutions* to *Equation 3*. But, to make our point above clear, the catch is that if $W \neq 0$, then $c_1$ and $c_2$ exist.

### 2.2 Repeated Roots

[1]Here's a brief refresher on forgotten high school math: Given the set of linear equations,

$$ax + by = c \tag{12}$$
$$dx + ey = f, \tag{13}$$

there's a nice way to find a solution for x and y using determinants,

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{W} \qquad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{W}, \tag{14}$$

where

$$W = \begin{vmatrix} a & b \\ d & e \end{vmatrix} \tag{15}$$

Note that y here doesn't have to be a solution to any differential equation. We're talking just boring ol'high school algebra here. This is just to show you how we arrive at our solution for $c_1$ and $c_2$ in the text—we are simply applying the method above to solving a set of linear equations.