# 1 First Order Linear Differential Equations

A regular old *linear function* is an equation that can be written in the general form,

$$\sum_i^n a_{1i} t_i^1 + a_0 t^0 = \sum_i^n a_{1i} t_i + a_0 = f, \tag{1}$$

where $f = f(t_1, ..., t_n)$ and $a_{1i},\ a_0$ are constants. The equation holds for $n$ dimensions. More familiarly, for $n = 1$, the slope-intercept $a_{11} - a_0$ form is recovered,

$$a_{11} t_1^1 + a_0 t^0 = a_{11} t + a_0 = f. \tag{2}$$

A *linear differential equation* is an equation that can be rewritten in the general form,

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{(n-1)} y}{dt^{(n-1)}} + \ldots + a_{n-1} \frac{dy}{dt} + a_n y = g(t), \tag{3}$$

where $a_i = a_i(t)$ and $y = y(t)$ where $i \in [0, n]$.[1] That is, both $a_i(t)$ and $y_i(t)$ must be functions *only* of $t$. Note that $a_i(t)$ may be any function of $t$, whether linear or nonlinear. The form is strikingly similar to that of a linear function. The laziest generalization from linear equations to linear differential equations is to use the derivatives $\frac{d^i y}{dt^i}$ as the linear variables. The $n^{th}$ derivative corresponds to the $n^{th}$ linear variable. (Because any differential equation must be an expression of two parameters, i.e., the function to differentiate ($y(t)$) and the differentiation variable $t$, the conditions on the constants $a_{1i}$ are now relaxed so that they can be functions of $t$.) Since mathematicians and nature tend to act according to the Principle of Least Action, i.e., the lazy person's ideal, that's why things are as they are.

*First order linear differential equations* are often referred to as

$$\frac{dy}{dt} = f(y, t), \tag{4}$$

where $f(y, t)$ is a *linear function* of $y$. Its dependence on $t$ can be linear or nonlinear.

## 1.1 Solution via Integration Factor

For convenience of the parameters involved in *the method of solution via integration factor* to first order linear differential equations, our favorite first order linear differential equation is often re-written as,

$$\frac{dy}{dt} + py = g, \tag{5}$$

where, as you might have caught on already, the convention of this text is to define $y = y(t)$ and $g = g(t)$ and $p = p(t)$. This form is just *Equation 4*, but with the function $f(y, t) = g - py$ expanded out explicitly.

Now, onward with the method!

[1] Recall that $y = \frac{d^0 y}{dt^0}$.

First, there's the business of putting things in the right form. To that end, let's multiply both sides of *Equation 4* by $\mu = \mu(t)$,

$$\mu\frac{dy}{dt} + \mu py = \mu g. \tag{6}$$

The idea of *the method of solution via integration factor* is to find some *integration factor* $\mu$ to rewrite the *Equation 6* as,

$$\frac{d\left(\mu(t)y(t)\right)}{dt} = \mu(t)g(t). \tag{7}$$

The most obvious (and lazy person's favorite) way of finding $\mu$ is to apply the product rule to the above derivative,

$$\frac{d\left(\mu(t)y(t)\right)}{dt} = \mu(t)\frac{dy(t)}{dt} + \frac{d\mu(t)}{dt}y(t), \tag{8}$$

thus making it look like the left side of *Equation 6*.

The goal is to make *Equation 7* replace *Equation 6*—our job is to find terms that make the two equations the same. Comparing this with *Equation 6*, we see from the second term in *Equation 6* that the following association can be made,

$$\mu p = \frac{d\mu(t)}{dt}. \tag{9}$$

This expression for $\mu p$ is a separable equation—we can blindly integrate it as if a baby-calculus problem. Dividing both sides by $\mu$ and multiplying both sides by $t$, we get,

$$pdt = \frac{d\mu}{\mu} \Rightarrow \int pdt = \int \frac{d\mu}{\mu} \Rightarrow \int pdt = \ln\mu \Rightarrow \mu(t) = e^{\int pdt}, \tag{10}$$

where we *don't care* about the constant of integration.
Thus, we have deduced the expression for the integrating factor,

$$\mu(t) = e^{\int pdt}. \tag{11}$$

Now, we can easily solve for $y(t)$ from *Equation 7* by multiplying both sides by $dt$ and integrating,

$$\mu(t)y(t) = \mu(t)g(t)dt \Rightarrow y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right). \tag{12}$$

And, the problem is solved! To recap, we have just done the following:

1. Rewrite the original first order equation in our favorite form:
$$\frac{dy}{dt} + py = g \Rightarrow \mu\frac{dy}{dt} + \mu py = \mu g \Rightarrow \frac{d\left(\mu(t)y(t)\right)}{dt} = \mu(t)g(t) \tag{13}$$
2. Find the integrating factor that does the job: $\boxed{\mu(t) = e^{\int pdt}}$
3. Solve for $y(t)$ therefore solving the differential equation: $\boxed{y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right)}$

Now, operationally, there are (at least) two ways to apply this method.

### 1.1.1 Method 1—Slacker's No-Brainer Way

To solve $\frac{dy}{dt} + py = g, \quad y(t_1) = y_1$, just do this:

1. Identify $p$ and $g$ in the above differential equation.
2. Find $\mu$ from *Equation 11* or $\mu(t) = e^{\int pdt}$.
3. Find the other terms required for the explicit statement of
$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right) \tag{14}$$
   (a) Find $\int^t \mu(s)g(s)ds$, where the variable $t$ has been replaced with $s$ in the integrand to avoid dependent limits.
   (b) Plug this in for the term $\int \mu(t)g(t)dt$ of *Equation 14*.
   (c) Apply the condition $y(t_1) = y_1$ to determine C.

### 1.1.2 Method 2

This method is good for understanding the derivation behind the method of integrating factors. As always, we start with identifying $p$ and $g$. Then,

1. Expand $\frac{dy\mu}{dt} = y^{'}\mu + y\mu^{'}$ to determine $\mu$. This recovers $\mu = e^{\int pdt}$.
2. Requote *Equation 7*,
$$\frac{d\left(\mu y\right)}{dt} = \mu g(t) \Rightarrow \mu y = \int \mu g(t)dt + C \tag{15}$$
Solve for y by dividing both sides by $\mu$.
3. Apply the condition $y(t_1) = y_1$ to determine C.