1 Exact Equations

An *exact differential equation* is something that can be written in this form:

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0 \tag{1}$$
$$= M(x,y)dx + N(x,y)dy = 0, \tag{2}$$

where, in the second line, we have defined M and N to be

$$M = \frac{\partial \psi}{\partial x} \tag{3}$$
$$N = \frac{\partial \psi}{\partial y}. \tag{4}$$

The form, $d\psi = 0$, has the form of an *exact differential.*[1]

For ψ to be a physical solution, i.e., continuous and nice and all that, we require $\boxed{\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}}$. But, this is just $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ by } \textit{Equation 4}}$. This is the *M-N condition for exactness.*

Alternatively, one can derive the M-N condition for exactness by a more illuminating method—one that also exposes the whole process for finding ψ:

Having written the above equations down, we can easily go about finding an implicit solution to our exact differential equation. In the process, we'd derive the M-N criterion via a way requiring a bit more work. (Watch closely; if you understand this, you'd never get confused about whether to integrate or differentiate M with respect to y or x or oh my!)

From *Equation 4*, we have $\frac{\partial \psi}{\partial x} = M$. We can easily integrate this equation by "multiplying" both sides by dx to get,

$$\psi = \int M dx + h(y) \tag{5}$$
$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\int M dx + h^{'}(y) = N \tag{6}$$
$$\Rightarrow h^{'}(y) = N - \frac{\partial}{\partial y}\int M dx \tag{7}$$
$$\frac{dh'(y)}{dx} = \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\int M dx\right) \tag{8}$$
$$\Rightarrow 0 = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}, \tag{9}$$

where instead of the usual constant C, we have the function $h(y)$, which is a *constant with respect to x.* (In general, the integral of a partial derivative involves a constant with respect to the variable you're *not* integrating with respect to.) Now, also from *Equation 4*, we have $\frac{\partial \psi}{\partial y} = N$, hence the equality on the second line. The third line solves for $h^{'}(y)$, and the fourth line takes the derivative with respect to x. Now, in the fifth line, we note that since $h(y)$ depends only on y, its derivative with respect to x is 0. Also, from the fundamental theorem of calculus, we have $\frac{d}{dx}\int M dx = M$. Basically, the derivative inverts the integral; thus, applying a derivative and then an integral amounts to doing nothing at all to the original function.

The last line arrives at the same result as above. The condition for exactness requires that the M and N's be set in the form,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y}}$$

Now, let's go about finishing the task we started above—that of finding our implicit solution, $\psi = c$.[2] The implicit solution looks like *Equation 5*. Using *Equation 6*, we can uniquely determine $h(y)$ by matching coefficients in N. That's basically all there is to finding our implicit solution.

[1]Exact differentials are useful in thermodynamics. The first law of thermodynamics relating work w, internal energy u, and heat q, is most usefully expressed in terms of an exact differential, namely, $dq = dw + du$. (Note that the sign convention here involves work done by the system as positive.)

[2]$\psi = c$ is a constant even if it might look like $\psi = x^2 + ysinc(x) = c$ since $d\psi = 0$.

1.1 Get an Integrating Factor to force an equation to be exact!

We've already done this problem for the Method via Integrating Factor case, where we had $y' + py = g \Rightarrow \frac{d\psi y}{dt} - \psi g = 0$—which, after being jumbled up by an integrating constant ψ, has the neato-form of an exact equation.

We can find an integrating factor by using the method of exact equations. We start with $Mdx + Ndy = 0$ and multiply both sides by our integrating factor, μ,

$$\mu(Mdx + Ndy). \tag{10}$$

Now, since this equation still has the same form as $d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$, associating $\frac{\partial\psi}{\partial x} = \mu M$ and $\frac{\partial\psi}{\partial y} = \mu N$, we can re-use our M-N condition above like this,

$$\frac{\partial\psi}{\partial x\partial y} = \frac{\partial\psi}{\partial y\partial x} \Rightarrow \frac{\partial\mu M}{\partial y} = \frac{\partial\mu N}{\partial x}. \tag{11}$$

If we assume that $\mu = \mu(x)$ (or $\mu = \mu(y)$, if the problem allows for an easier integrating factor that way), we can apply the product rule to find an expression for μ,

$$\frac{\partial\mu M}{\partial y} = \frac{\partial\mu N}{\partial x} \tag{12}$$

$$\mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{d\mu}{dx} \tag{13}$$

$$\mu(M_y - N_x) = N\frac{d\mu}{dx}. \tag{14}$$

The partial derivative becomes a total derivative in line two because we've chosen $\mu = \mu(x)$; but, in general, you can choose $\mu =$ *anything you desire*—be it Airy Apple or Bessel Banana: anything. Note that there is no need to memorize the last formula, since you can derive it from the M-N condition.

1.2 The Steps

If the above goes over your head, you can just blindly memorize the steps below. (Note that I've numbered the steps of my posted solutions in the Boyce&diPrima solutions website according to this convention.)

1. Find M and N in the equation $Mdx + Ndy = 0$. *Example:* $(x^2 + y)dx + (x/y + \sin(x))dy = 0 \Rightarrow M = x^2 + y \quad N = x/y + \sin(x)$
2. Check for exactness using the M-N condition, $M_y = N_x$. If the equality holds, then the equation is *exact as is. Example:* $ydx + xdy = 0$ *has* $M_y = 1 = M_x$*, and thus is exact.* (If it's exact, you can skip the rest of this step and its sub-step and go on to the next.) If not, then you can try to find an integrating factor to make it exact.
   (a) You can *find an integrating factor* by using a modified M-N condition, $(\mu M)_y = (\mu N)_x$. Please note that we're taking the derivative of the whole quantity (μM) or (μN). This means that the product rule is key.
   (b) Usually, you're allowed to choose $\mu = \mu(x)$ or $\mu = \mu(y)$; that is, μ is usually just a function of *either* x or y. Thus, if you choose $\mu = \mu(x)$ the derivative becomes just $(\mu M)_y = \mu M_y$.
   (c) *Example:* $(x^2 + y)dx + x^5dy = 0$ *is not exact, as is since* $M_y \neq N_x$*, as you can quickly check. However, we can find an integrating factor such that* $\left(\mu(x^2 + y)\right)_y = \left(\mu x^5\right)_x$ *is an equality.* Using $\mu = \mu(x)$ and chunking out the product rule, you should find that $\mu = 5\mu x^4 + \mu' x^5 \Rightarrow \frac{d\mu}{dx} = \mu\frac{1-5x^4}{x^5}$, which is a separable equation that you can easily integrate.
3. Now, just integrate, $\psi = \int Mdx + h(y)$, where $h(y)$ is a constant with respect to x. Then, take the derivative and equate coefficients with N, $\frac{\partial\psi}{\partial y} = N$ to find $h'(y)$. The final solution is just $\psi = c$. (*Example: in the equation* $ydx + xdy = 0$*, we have* $M = y \quad N = x$*. Thus,* $\psi = \int ydx + h(y) = xy + h(y)$*. But,* $\psi_y = x + h'(y) \equiv N$*. Since* $N = x$*,* $h' = 0 \Rightarrow h = constant$*. Thus,* $\psi = x = c$*.*) Note: This applies even for the case of integrating factors if you make the substitution $M \to \mu M$ and $N \to \mu N$.